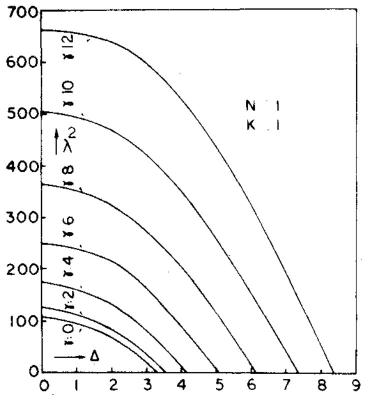

Fig. 1 Values of frequency and buckling parameters for a simply supported beam.

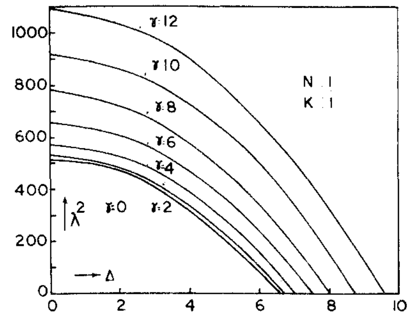

Fig. 2 Values of frequency and buckling parameters for a fixed-fixed beam.

Substituting Eq. (24) in Eq. (12) and performing the Galerkin method of integrating the resulting expression over the whole length of the beam, the expression for the frequency parameter λ is obtained as

$$\lambda = 2\{(N^2\pi^2/3)(4N^2\pi^2+K^2-\Delta^2)+\gamma^2\}^{1/2} \tag{25}$$

In arriving at Eq. (25), only one term of the infinite series of Eq. (24) is utilized. Eq. (25) gives an upper bound for the natural frequency parameter λ for the fixed-end beam as it is obtained by the approximate method due to Galerkin. But it is quite handy and can give values within engineering accuracy.

By putting $\lambda = 0$, and $N = 1$, in Eq. (25), the expression for the buckling load parameter Δ_{cr}, for the clamped beam can be obtained as

$$\Delta_{cr}^2 = [4\pi^2+K^2+(3/\pi^2)\gamma^2] \tag{26}$$

For the simply supported-fixed beam, the approximate expressions obtained, for the fundamental frequency parameter λ ($N = 1$), and buckling load parameter Δ_{cr}, by the Galerkin method assuming a power series of the type

$$X(Z) = \sum_{i=0}^{4} a_i Z^i \tag{27}$$

are

$$\lambda = [238.739+11.3686(K^2-\Delta^2)+4\gamma^2]^{1/2} \tag{28}$$

and

$$\Delta_{cr}^2 = [21+K^2+0.352\gamma^2] \tag{29}$$

Out of the five constants a_i ($i = 0, 1, 2, 3, 4$) in Eq. (27), four constants as ratios of the fifth constant can be obtained by utlizing the boundary conditions of Eq. (10) for this case, and the fifth arbitrary constant cancels out in the Galerkin integral.

In the limiting case of the absence of elastic foundation, i.e., $\gamma = 0$, and the compressive load, $\Delta = 0$, all the approximate expressions are observed[9] to be in complete agreement with those derived previously by Gere[3] and Timoshenko and Gere.[7]

Conclusions

Results for the torsional frequency parameter λ, for the first mode ($N = 1$), for the simply supported and fixed beams, obtained from Eqs. (22) and (25) are plotted in Figs. 1 and 2 respectively, for various values of foundation parameter γ and load parameter Δ. The warping parameter is kept at $K = 1$. The values of the critical buckling loads for various values of γ can also be obtained from the graphs for $\lambda = 0$ (i.e., on the axis on which Δ is taken). When the axial load is not present the values of the frequency parameter λ for various values of γ can be obtained for values of $\Delta = 0$ (i.e., on the vertical axis on which λ is plotted). The combined influence of the foundation parameter γ and the load parameter Δ can be observed from the graphs, to be opposing each other. Independently, as the load parameter Δ increases, the frequency parameter λ drops to zero. In the absence of the axial load, the frequency parameter λ increases for increasing values of the foundation parameter γ. Hence the combined influence is the superimposition of the individual effects on the frequency of vibration.

It can be observed from Eqs. (22) and (25) that the influence of foundation parameter γ decreases for increasing values of N (i.e., for higher modes). It is interesting to see from Eq. (22) that for the simply supported beam, for the limiting condition $\gamma = 0.5N\pi\Delta$, the combined influence of elastic foundation and axial compressive load becomes zero. In the case of clamped beam this limiting condition, from Eq. (25), is $\gamma = 0.574N\pi\Delta$.

References

[1] Hatenyi, M., *Beams on Elastic Foundation*, University of Michigan Press, Ann Arbor, Mich., March 1946.

[2] Vlasov, V. Z. and Leont'ev, U. N., "Beams, Plates and Shells on Elastic Foundations," translated from the Russian, Israel Program for Scientific Translations, Jerusalem.

[3] Gere, J. M., "Torsional Vibrations of Beams of Thin-Walled Open Section," *Journal of Applied Mechanics*, Transactions of the ASME, Dec. 1954, pp. 381–387.

[4] Aggarwal, H. R. and Cranch, E. T., "A Theory of Torsional and Coupled Bending Torsional Waves in Thin-Walled Open Section Beams," *Journal of Applied Mechanics*, Transactions of the ASME, Vol. 34, 1967, pp. 337–343.

[5] Krishna Murty, A. V. and Joga Rao, C. V., "General Theory of Vibrations of Cylindrical Tubes," *Journal of the Aeronautical Society of India*, Vol. 20, No. 1, 1968, pp. 1–38; No. 4, pp. 235–258.

[6] Kameswara Rao, C., Apparao, K. V., and Sarma, P. K., "Effect of Longitudinal Inertia and Shear Deformation on the Torsional Frequency and Normal Modes of Thin-Walled Open Section Beams," *Journal of the Aeronautical Society of India*, to be published.

[7] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[8] Christiano, P. and Salmela, L., "Frequencies of Beams with Elastic Warping Restraint," *Journal of the Structural Division, Proceedings of the ASCE*, Vol. 97, No. ST6, June 1971, pp. 1835–1840.

[9] Appala Satyam, A., "Torsional Vibrations and Stability of Thin-Walled Beams of Open Section Resting on an Elastic Foundation," M.E. thesis, 1974, Department of Mechanical Engineering, Andhra University, Waltair, India.

Vibration—Stability Relationships for Conservative Elastic Systems

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Introduction

LURIE[1] observed that, for linearly elastic systems, the square of the lateral frequency is "practically" linearly related to the end thrust. According to Southwell's theorem[2] this straight

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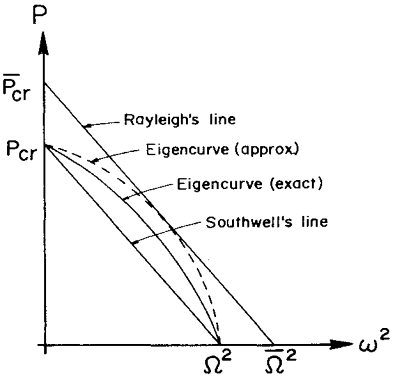

Fig. 1 Load vs frequency square curves based on exact free vibration and buckling modes.

line is in fact a lower bound. Recently Singa Rao and Amba-Rao[3-5] investigated how far the actual relationship deviates from the lower bound, using Rayleigh-Ritz method. In this paper an alternate procedure to determine the measure of the deviation is presented. This procedure, in contradistinction to Refs. 3-5, does not require the plotting of the load vs frequency square curve.

Elastic System and Eigencurve

Consider the lateral vibration of an undamped, linearly elastic system subjected to stationary, conservative, compressive inplane loads. Let the lateral deflection

$$w(x_j, t) = e^{i\omega t}u(x_j) \tag{1}$$

where ω is the frequency of vibration, x_j are the spatial coordinates, t is the time, and $i = (-1)^{1/2}$. The total energy of the system is

$$E = V - PW - \omega^2 T \tag{2}$$

where the terms on the right-hand side represent the internal strain energy, work done by the inplane loads and the kinetic energy, respectively. V, W, and T are quadratic functionals in the displacement function $u(x_j)$. P is the load parameter.

This is a two-parameter (load P and frequency square ω^2) eigenvalue problem. The natural frequencies Ω_k and the free vibration eigenmodes $\phi_k(x_j)$ are obtained when $P = 0$; the buckling loads $P_{cr,k}$ and the buckling modes $\psi_k(x_j)$ are obtained when $\omega^2 = 0$. In this Note our attention will be restricted to the fundamental frequency ω_1 and the fundamental buckling load $P_{cr,1}$ only. Henceforth the subscript 1 will be dropped and they will be referred to as ω and P_{cr}.

A typical load vs frequency square curve (eigencurve) is shown in Fig. 1. The straight line joining Ω^2 and P_{cr} is a lower bound to the eigencurve in the first quadrant of the $P-\omega^2$ plane†[2] and may be called the "Southwell line." If the free vibration mode is same as the buckling mode, i.e., $\phi(x_j) \equiv \psi(x_j)$, the eigencurve coincides with this line.[1] Even otherwise, according to Lurie,[1] the deviation of the eigencurve from the Southwell line is very small.

Rayleigh-Ritz Method

In many problems it is not possible to obtain the eigencurve exactly. In such cases an approximate eigencurve may be obtained by a two-term Rayleigh-Ritz solution. Let the approximate deflection function be expressed in terms of the exact free vibration mode and the buckling mode; i.e.,

$$\bar{u}(x_j) = a_1\phi(x_j) + a_2\psi(x_j) \tag{3}$$

where $\bar{u}(x_j)$ is the approximate deflection function and a_1, a_2 are unknown coefficients.

Substitution of Eq. (3) in Eq. (2) and the minimization of the energy gives the values of a_1, a_2, and ω for different values of P, and the approximate eigencurve can be plotted (Fig. 1). This will be an upperbound for the exact eigencurve.

† All discussions in this Note are restricted to this quadrant only.

Modified Rayleigh Method

The present method is a modification of the well known Rayleigh's method. Instead of a two-term Rayleigh-Ritz solution, let us find the (one-term) Rayleigh quotient for the system, using an approximate deflection function

$$\bar{u}(x_j) = b_1[\phi(x_j) + k\psi(x_j)] \tag{4}$$

where k is a prescribed number and b_1 is the "unknown" coefficient. For different values of k, we obtain different Rayleigh quotients.

Substitution of Eq. (4) in Eq. (2) and the minimization of the energy gives an algebraic equation of the form

$$F + PG + \omega^2 H = 0 \tag{5}$$

where

$$F = f_1 + f_2k + f_3k^2$$
$$G = g_1 + g_2k + g_3k^2 \tag{6}$$

and

$$H = h_1 + h_2k + h_3k^2$$

in which $f_1, f_2, \ldots h_3$ are functionals of $\phi(x_j)$ and $\psi(x_j)$.

Equation (5) is a straight line, which may be called the "Rayleigh line," in the P vs ω^2 plane, and we get one such line for each value of k. When k is equal to (a_2/a_1) of the two-term Rayleigh-Ritz solution [Eq. (3)], the Rayleigh line will touch the approximate eigencurve at the corresponding load. So each point of the curve has a corresponding k value and a Rayleigh line. These Rayleigh lines cannot intersect the curve at more than one load, since if they do, it means that the system has the same eigenmode for more than one load; and then the eigencurve should be a straight line, and the exact eigencurve, approximate eigencurve, the Rayleigh line, and the Southwell line will all coincide. So, if the eigencurve is not a straight line, the Rayleigh line is a tangent to it.

Slope of the Rayleigh line is

$$dP/d\omega^2 = \theta = -H/G \tag{7}$$

Approximate critical load obtained from the Rayleigh line is

$$\bar{P}_{cr} = -F/G \tag{8}$$

and the approximate natural frequency square is

$$\bar{\Omega}^2 = -F/H \tag{9}$$

Deviation of the Eigencurve from the Southwell Line

The deviation of the two-term Rayleigh-Ritz eigencurve is a maximum from the Southwell line, when it has a tangent parallel to the above line (see Fig. 1). Since the two-term Rayleigh-Ritz curve is an upperbound to the exact eigencurve, this deviation is also an upper bound for the deviation of the exact eigencurve. Also, the above tangent is an upper bound to the eigencurve. This "upper bound line" can be obtained by equating the slope of the Southwell line $s(= -P_{cr}/\Omega^2)$ to θ; i.e.,

$$s = -(h_1 + h_2k + h_3k^2)/(g_1 + g_2k + g_3k^2) \tag{10}$$

Simplifying, we get

$$k^2(h_3 + g_3s) + k(h_2 + g_2s) + (h_1 + g_1s) = 0 \tag{11}$$

Let $k_1{}^*$ and $k_2{}^*$ be the roots of Eq. (11). Substituting $k_1{}^*$ and $k_2{}^*$ in Eq. (9), two values of $\bar{\Omega}^2$ can be obtained. The one giving the lower value, $\bar{\Omega}^{*2}$, is denoted by k^*, and this corresponds to the required tangent. If the other value of k gives a value lower than $\bar{\Omega}^{*2}$, then the corresponding Rayleigh line will intersect the approximate eigencurve and this, as already discussed, is inadmissible. Therefore

$$\bar{\Omega}^{*2} = -(f_1 + f_2k^* + f_3k^{*2})/(h_1 + h_2k^* + h_3k^{*2}) \tag{12}$$

So an upper bound for the deviation of the eigencurve from the Southwell line is

$$\Delta = \bar{\Omega}^{*2} - \Omega^2 = \Omega^2(\bar{\Omega}^{*2}/\Omega^2 - 1) \tag{13}$$

For some systems, the exact vibration mode and buckling mode may not be available. In such cases, approximate modes

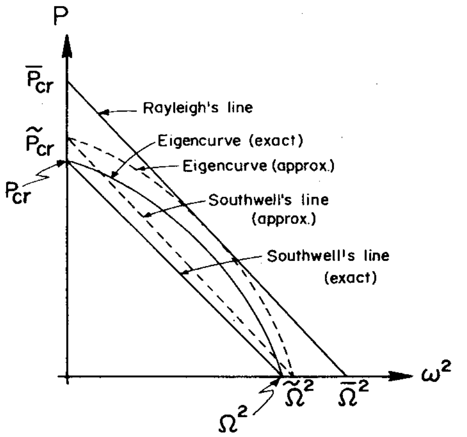

Fig. 2 Load vs frequency square curves based on approximate free vibration and buckling modes.

may be used in the procedure just discussed. However, the straight line joining the approximate natural frequency square $\tilde{\Omega}^2$ and approximate buckling load $\tilde{P}_{cr}$, obtained using the approximate modes, will no longer be a lower bound (Fig. 2); but the Rayleigh line parallel to this line will still be an upper bound.

Example

Consider the vibration and stability of a uniform clamped-clamped beam of length l, subjected to compressive end forces P. Exact values of the fundamental natural frequency and buckling load are

$$\Omega^2 = 500.564 EI/\mu l^4$$

and (14)

$$P_{cr} = 4\pi^2 EI/l^2$$

where E = Young's modulus of elasticity, I = moment of inertia of the cross section, and μ = mass per unit length. The vibration and buckling modes are

$$\phi(x) = \left(\cosh\frac{\lambda x}{l} - \cos\frac{\lambda x}{l}\right) - \beta\left(\sinh\frac{\lambda x}{l} - \sin\frac{\lambda x}{l}\right) \tag{15}$$

and

$$\psi(x) = [1 - \cos(2\pi x/l)]$$

where $\lambda l = 4.73004$ and $\beta = 0.9825022$. The total energy of the system, similar to Eq. (2), is

$$E = \int_o^l (EIu''^2 - Pu'^2 - \mu\omega^2 u^2)\,dx \tag{16}$$

where $'$ denotes differentiation with respect to x. The coefficients $f_1, f_2 \ldots h_3$, defined in Eq. (6), are obtained using Eq. (16) as

$$\begin{aligned} f_1 &= EI\int_o^l \phi''^2\,dx, & f_2 &= 2EI\int_o^l \phi''\psi''\,dx & f_3 &= EI\int_o^l \psi''^2\,dx \\ g_1 &= -\int_o^l \phi'^2\,dx, & g_2 &= -2\int_o^l \phi'\psi'\,dx, & g_3 &= -\int_o^l \psi'^2\,dx \\ h_1 &= -\int_o^l \phi^2\,dx, & h_2 &= -\int_o^l \phi\psi\,dx, & h_3 &= -\int_o^l \psi^2\,dx \end{aligned} \tag{17}$$

Calculations as indicated in the previous section yield

$$\Delta = 0.00836\,\Omega^2 \tag{18}$$

i.e., an upperbound for the deviation of the eigencurve from the Southwell line is only 0.836% of the natural frequency square.

Conclusions

The method developed in this Note is applicable to any multiple parameter eigenvalue problem, for example, buckling of an elastic structure under multiple independent loadings. For an N-parameter eigenvalue problem, we will obtain two parallel $(N-1)$ degree hyper-surfaces as upper and lower bounds, instead of two parallel lines in the case of the two parameter eigenvalue problem.

References

[1] Lurie, H., "Lateral Vibrations as related to Structural Stability," *Journal of Applied Mechanics*, Vol. 19, 1952, pp. 195–204.

[2] Temple, G. and Bickley, W. G., *Rayleigh's Principle*, Oxford University Press, London, 1933.

[3] Singa Rao, K. and Amba-Rao, C. L., "Lateral Vibration and Stability Relationship of Elastically Restrained Circular Plates," *AIAA Journal*, Vol. 10, No. 12, Dec. 1972, pp. 1689–1690.

[4] Singa Rao, K. and Amba-Rao, C. L., "Comment on 'Lateral Vibration and Stability Relationship of Elastically Restrained Circular Plates'," *AIAA Journal*, Vol. 11, No. 7, July 1973, p. 1056.

[5] Flax, A. H., "Comment on 'Lateral Vibration and Stability Relationship of Elastically Restrained Circular Plates'," *AIAA Journal*, Vol. 11, No. 11, Nov. 1973, pp. 1599–1600.

New Device for Skin-Friction Measurement in Three-Dimensional Flows

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Nomenclature

D = outer diam of probe tube
d = inner diam of probe tube
p_o = pressure measured by the total head tube of the probe
p_c = pressure measured by the chamfered tube of the probe
p_s = wall static pressure
u_τ = friction velocity ($= [\tau_w/\rho]^{1/2}$)
v = kinematic viscosity
ρ = fluid density
τ_w = wall shear stress

I. Introduction

SEVERAL experimental methods are available for measurement of skin-friction. A brief review of these methods is given by Brown and Joubert.[1] A class of these methods is based on the principle of similarity of flow about obstacles. These methods rely on the observation that the flow near the wall is governed by the "wall variables." According to this principle, the velocity field about an obstacle immersed entirely in the "law of the wall" region is completely determined by the wall variables. The wall variables are surface friction, τ_w, density and kinematic viscosity of fluid, ρ, v, and a characteristic length of obstacle, l. Dimensional reasoning gives that any pressure difference is given by (for details, see Brown and Joubert,[1] p. 742)

$$\Delta p l^2/\rho v^2 = f[\tau_w l^2/\rho v^2] \tag{1}$$

This function can be determined experimentally for a given device. Many designs of devices based on this principle are available, such as the Preston-tube,[2-4] Sublayer fence,[5,6] Stanton tube or razor-blade technique,[7,8] Static hole pair[9] etc. It is to be noted here that in all of these methods one of the pressures to be measured is invariably the wall pressure. By contrast, the new device given in this work does not require a wall tapping at the point of measurement. Provision of wall static

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